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ON THE EXCITATION OF OSCILLATIONS BY PARAMETER VARIATION (HETERO-PARAMETRIC EXCITATION)

By V.A. LAZAREV

Abstract: An experimental examination of hetero-parametric excitation — based on the principles of L. Mandelstam and N. Papaleski — was obtained by the construction of an alternating-current parametrical generator.
Additionally, this generator had, as part of its construction, periodic mechanical undulation of the self-induction in order to observe the effect of their modification. Among the various factors that were surveyed included: logarithmic decrement of damping, depth of self-induction modulation, windage loss and relaxation time of the disk, and the amplitude conditions of quantity, excitation, and magnitude. The experiments revealed that there are two varying conditions of stationary (steady state) amplitude.
This is consistent with the theory developed by L. Mandelstam and N. Papaleski, which proves to be as true quantitatively as qualitatively (within the limits of accuracy of the observations of the experimental results described here).

Introduction

The present work represents an experimental foray into research on the phenomena, including examining the appearance of oscillation excitation levels taking place by means of the periodic modification of self-induction in the oscillating system (hetero-parametric excitation) and a qualitative verification of the theoretical results given by L.I. Mandelstam and N.D. Papaleski. Before communicating the experimental data acquired, it is probably expedient to briefly compare the principal conclusions developed by the above-named authors and to also specify those theoretical assumptions upon which it was formulated. As the work of L.I. Mandelstam and N.D. Papaleski has shown,[1] in electrically oscillating systems in which there are no special radiating electric or magnetic fields it is possible to excite and have long term electrical oscillations by means of (either electrical or mechanical) periodic modifications of the self-induction or capacitance of the systems. Thus, the following are the basic conditions for such excitation (which has received the label of hetero-parametric excitation):

a) It is necessary to modify the frequency controlling parameters at approximately twice greater than the system's average fundamental frequency.

b) It is necessary to observe certain parameter relations between the magnitude of the parameter modifications (stipulating the magnitude input into system from the external fields) and the magnitude of the average characteristic logarithmic decrement of damping (such that the input power into the system exceeds the losses within in it).

[1]L.I. Mandelstam and N.D. Papaleski. See above

More accurate information about these requirements is gained by reviewing the so-called unstable solutions of linear differential equations with periodic coefficients, in which (considering a modification of self-induction) the oscillating condition of the system reduces to,

$$L = L_0\left(1 + m\sin 2\omega t\right) \quad . \tag{1}$$

That leads to the following differential equation

$$L_0 \frac{d}{dt} i \left(1 + m\sin 2\omega t\right) + Ri + \frac{1}{C}\int i\, dt = 0 \quad . \tag{2}$$

It is possible here to formulate more precisely the [necessary] requirements as follows:

$$m > \frac{2}{\pi}\delta\,, \tag{3}$$

$$1 + \sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}} \quad \geq \quad \frac{\omega_0^2}{m^2} \quad \geq \quad 1 - \sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}} \quad . \tag{4}$$

Here $\delta = \dfrac{R}{L_0\omega}$ is the "average" logarithmic decrement of the natural oscillations of the system, 2ω is the frequency of the parameter change, $\omega_0 = \dfrac{1}{\sqrt{L_0 C}}$ is the "average" natural frequency of the system, and $m = \dfrac{L_{\max} - L_{\min}}{L_{\max} + L_{\min}}, \left(or \quad m = \dfrac{C_{\max} - C_{\min}}{C_{\max} + C_{\min}} \right)$ is the relative quantity of the parameter change (the so-called "depth of modulation" of the parameter). Requirement (4) can be written in a slightly different form. If we insert "detuning", namely:

$$\xi = \frac{\omega^2 - \omega_0^2}{\omega^2} \quad , \tag{5}$$

where

$$\sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}} \quad \geq \quad \xi \quad \geq \quad -\sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}}\,. \tag{6}$$

then, the quantity of the hetero-parametric field excitation, as expressed through ξ, is equal to

$$\xi_1 - \xi_2 = \frac{\omega_{(1)}^2 - \omega_{(2)}^2}{\omega^2} = 2\sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}} \quad . \tag{7}$$

Equations 3, 6 and 7 are sufficient to guarantee the occurrence of oscillations. It is also necessary to note that, in the beginning of the process the vibration amplitude accrues approximately under the law:

$$A = \frac{1}{e^2}\left(\sqrt{\frac{m^2}{4} - \xi^2} - \frac{\delta}{\pi} \right)\omega t \quad . \tag{8}$$

In order to derive the magnitude of the stationary amplitude it is necessary to consider the influence of the factors limiting unbounded amplitude growth in systems for which such growth is impossible without such growth being introduced. Such factors depend upon the magnitude of resistance included in the system, i.e. - a circuit of incandescent lamps or self-induction in iron-core coils (such as that taking place in the present research, which uses a nonlinear relation between the magnetizing current and magnetic induction). On account of these factors, the mathematical problem of determining the stationary amplitude leads to the solution via nonlinear differential equations. In the specific theoretical analysis, the approximate solution of this problem is given for some simple cases; for example when there is an expressed relation between the current and the magnetic flux, or when a cubic parabola is given:

$$\Phi_2 = \alpha\, i + \gamma\, i^3 \tag{9_1}$$

or, as based on the citation below,[2]

$$\Phi_2 = \gamma\ arctg\ k\,i + \alpha\, i \quad . \tag{10_1}$$

It is thus supposed that only the linear part of the self-induction system is exposed to an exterior periodic modification and looks like the differential equation expressed as

$$\frac{d\Phi}{dt} + Ri + \frac{1}{C}\int i\,dt = 0 \tag{11}$$

where

$$\Phi = L_0(1 + m\sin 2\omega t)i + \gamma\, i^3 \tag{9_2}$$

and

$$\Phi = L_0(1 + m\sin 2\omega t)i + \gamma\, arctg\, ki - \gamma\, ki \quad . \tag{10_2}$$

For a quadratic voltage amplitude on the condenser, X^2, the following approximate values emerge. In the first case

$$\frac{3}{4}\mu X^2 = \xi + \sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}} \quad when \quad \gamma < 0 \tag{12_1}$$

where

$$\frac{3}{4}\mu X^2 = -\xi + \sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}} \quad Where \quad \gamma > 0\ . \tag{12_2}$$

$$\left(\mu = \frac{\gamma\omega^2}{L_0}\right)$$

In the second case

$$X^2 = \frac{8g}{k^2} * \frac{2\xi + \sqrt{m^2 - 4\delta^2}}{\left[2(y - \xi) - \sqrt{m^2 - 4\delta^2}\right]^2} \tag{13}$$

where

$$g = \frac{k\gamma}{L_0} \quad .$$

Examination of the stability of these solutions has shown that they are inconvertible for all values ξ, starting from

$$\xi \geq -\sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}} \quad \left(where \quad \xi \leq \sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}}\right), \tag{14}$$

up to that quantity ξ for which estimates were made considering the order of smallness that will hold good for $m, \delta, and\ \mu$.

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[2] From the work of V.P. Gulyaeva and V.V. Megulena, Limited release.

In particular the possible incontrovertible values of $\xi \geq \sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}}$, i.e. ($\Phi$ in Eq. 9) for limits of the excitation field. The phenomenon differs here in that a reduction (in the frequency hysteresis) is possible.

An examination of hetero-parametric excitation is given below that is based on an original manner of varying the self-induction of the vibrating system that is different from that of L.I. Mandelstam and N.D. Papaleski. The present apparatus consists in periodically introducing the metal masses of a toothed wheel into an array of coils that constitute the self-induction coils of the oscillating system.[3] Here, the alteration of self-induction results from a modification of the magnetic energy of the system due to Foucault's magnetic field currents occurring in the moving toothed wheel. This specific case differs from the idealized models for an oscillating system with variable self-induction considered above in that here the self-induction of the system is not immediately effected, but varies with changes in the system's metal masses where it is possible to incorporate them into a closed coil (a toroidal-ring), and where the self-induction and resistance (which do not remain at constant levels during the mechanical motion and electrical driving current) are "modulated" simultaneously with the mutual induction. On the one hand, the rigorous theory of this case represents a series of greater difficulties (and is not developed yet). On the other hand, it is possible to obtain a first approximation to reduce the effort required to analyze this case as one of periodic modification of self-induction. So, we shall start by reducing the above formulas.

Description of the Apparatus

For carrying out the hetero-parametric excitation experiments, we constructed an experimental apparatus in which the parameter L was varied by an exterior mechanical force. For this purpose, an original machine for testing hetero-parametric excitation oscillations was built. In this case, the machine's stator frames were made of a dry Karelian birch wood, and consisted of two halves with the miniscule clearance of a millimeter between them. In this case, 8 series-connected coils with identical distances from each other were arranged around a toothed disk of 30 cm diameter. There were, in total, 126 turns (wound with wires of 1.56 mm diameter on special laminated iron stator cores) on each of the induction coils. In figures 1 and 2 we show the stator in the side view and the coil separately. A 30 cm diameter duraluminum disk was placed within the stators, having a clearance of 6 mm, with a 3 mm thickness. 8 teeth were cut out upon the periphery (the same as the number of coils), which were equally spaced around the circumference. The sizes of the teeth were defined by the cross section of the stator cores. Their length was 5 cm, and their breadth 4 cm. In figure 3 the disk is shown separately. Hence the teeth have been constructed so that they could be fitted bodily inside the housing where the coils are located, and fitted in the air gaps between the coils in the air gap in the individual stator cores. Figure 4 shows the machine without its upper stator. Figure 5 is a general view of the machine with a driver part given. The entire system was set in motion by a 3-phase current type of motor manufactured by the Electric Power Factory.

[3] From L.I. Mandelstam and N. Papaleski, loc. cit. page 5.

Using 220/127 VAC with a power of 4.5 kW, we achieved about 1430 revolutions per minute with this motor and, additionally, we attached a 30 kilogram flywheel to the motor's shaft. The motor has been connected with the toothed disk of the machine through a 10-fold speed reducer running in reverse as a speed "increaser", so that the speed of the disk reached up to 14,300 revolution per minute.

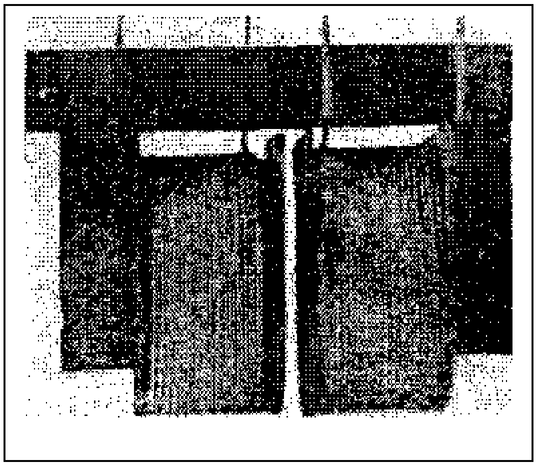

Figure 1: Single Coil with Laminated Core

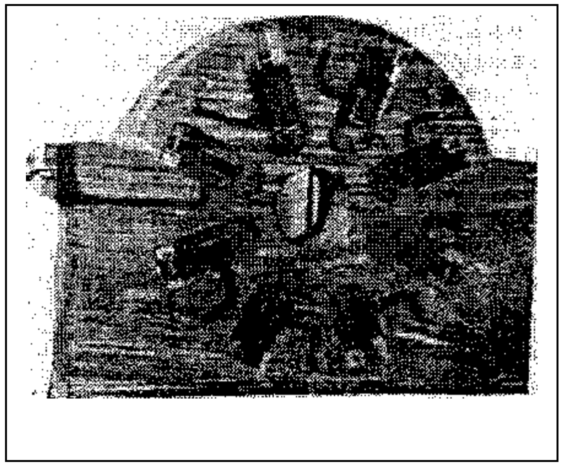

Figure 2: Assembled Stator Coils in Frame

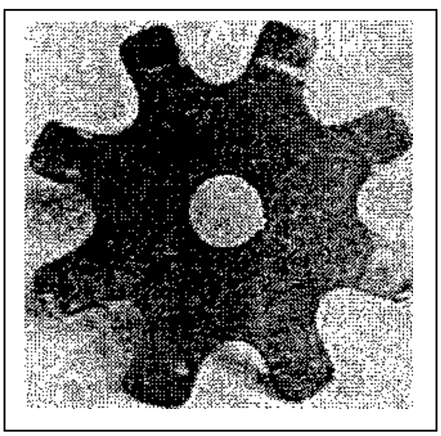

Figure 3: Toothed Disk Rotor

Figure 4: Rotor Assembled in Bottom Half of Stator

We had special ball bearings cast for the disk's shaft and applied forced lubrication. As a demonstration of the low friction of these bearings, when the disks were struck they would move easily.[4]

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[4] It is necessary to note the critical and invaluable participation of the laboratory's mechanic-designer, M. I. Resnika, in the development of these laboratory installations.

When the disk rotates, there is a common reactance acting between the teeth and the laminated core coils while the reactance among the stator coils varies, and, consequently, the self-inductance of the system varies. When the teeth are in the air gap of the coil's laminated cores there is a minimum of self-induction, and, contrastingly, when the teeth are between the coils there is a maximum of self induction. When the motor with the speed increaser is operating the toothed disk at a speed of N=14,300RPM, the periodicity of the self induction of the stator varies near the frequency n = $\frac{8N}{60}$ = 1900 hertz (or close to that). Hence, having attuned the system to the median frequency of 950 hertz and having the power available to drive the system, we raised the oscillations frequency to 950 hertz. And, we could vary the drive power of the excited oscillations, as we desired, in a controlled fashion that reached up to 4 kilowatts.

Investigation of the Excitation Requirements

Figure 5.

First of all, the conditions for the occurrence of the fluctuations have been previously investigated for hetero-parametric excitation. These conditions are given by the formulas 3, 6, 7 and 8.

For a check of Equation (3) in the above described installation, the depth of modulation m was measured as well as the logarithmic decrement of the system. All measurements were made via the Vitstona bridge at a frequency of 950 hertz. For the self-induction measurement, the bridge's output used a Hartman and Brown meter with a resistance of 100,000 ohms and a Siemens and Galska self-induction reference of 0.1 Henry, where the natural frequency of the equipment was defined via the corresponding bridge as well. In this case the measured output itself represented the resonant circuit, along with the three other ohmic resistances. During a contour moment resonance, the bridge can be balanced cleanly by ohmic resistance. Loss data was taken at the contour simultaneously by both measuring self-induction and capacity, while introducing to the bridge an additional resistance reference. Overall losses of the contour developed in two parts: R_C (aggregate capacity losses) and R_L (aggregate losses via self-induction). The aggregate loss data, via self-induction, represents the total loss in the winding, the iron cores used, and the disk. It then becomes especially necessary to address the question of losses originating in the disk. The role of the disk consists of putting energy into the system. The Foucault currents along the disk's teeth, which interact with the currents of the coils, causes the teeth to be pushed out from the group of coils. In other words, a tooth "standing" in between coils does not convert energy, and pushing "against" the coils is labile and does convert energy. Hence while the teeth are standing in between the coils, the system possesses an energy minimum, while the in the second position a maxima. When transitioning from the first stance (in between coils) to the second, we expend mechanical energy, and obtain operation E_1. And backward, at the next transition, from a labile equilibrium in a non-convertible system would be to return energy E_2. The total amount of energy that had been acquired during the system's transition from the first equilibrium stage to that left by the second stage will be equal to

$$\Delta E = E_1 - E_2 > 0 \tag{15}$$

Equation (15) is enacted due to the establishment of a corresponding phase of the position between coil current and a standing disk in space. On the other hand, the disk is a main constituent of the entire vibrating system, and we should not import to it greater current losses, as the requirement of self-excitation (3) otherwise will not be satisfied. In other words, the relaxation time of the disk should not be less than a certain [specified] quantity.

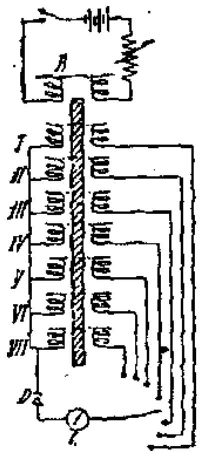

Figure 6.

This requirement boils down to the Foucault currents induced with the disk, which could not be damped during the allotted time, for the required tooth transition from the given coil to the next nearest one. Otherwise, it would be difficult to present the process of increase. The disk should be the current carrier from one coil to another and thus induce in the coils additional impulses, which are the origin of any further increases in oscillation.

The fact is that these additive impulses are induced by a disk among coils, and should be in phase with the currents already extant in the coils. This demands that the frequency of the contour is in a certain specific relation to the frequency of modulation of the self-induction, where a frequency relation of 1:2 is especially effective.

It was necessary to measure observationally in order to find out what the standard requirement concerning relaxation time is as displayed on a disk. However, owing to the necessity of the special synchronization of impulses, to yield such measurement data above a toothed disk turned out to be inconvenient and we were restricted to only our experiments with continuous disks.

Definition of Relaxation Time in Continuous Disks

We have produced measurement data of the relaxation time in a continuous (non-toothed) rotating duraluminum disk. Our experience is put as follows.

For the toothed disk in the previously described parametric machine we substituted a continuous disk, of the same diameter, made from duraluminum but with a thickness of 3 mm. The 8-coiled stator, which can be seen separately in Figure 2, had noticeable separations between themselves. The continuous disk rotated without difficulty in the inter-polar space between the coils. One of the coils **A** (Figure 6) was moved, over a major time interval (in one to two minutes), via identical impulses of direct current.

The quantity of impulses crept up such that, at the final submission at coil A during the immobilization of the disk, galvanometer **Γ**, which is part of a network of the nearest next coil (for us, the millivoltmeter of the Drains firm on 17 mV with its own period of 2 seconds has been included, with an internal resistance of 325 ohms). This was fed through detector **P** which did not prove difficult. The identification of the impulse could be easily attained using a closure current on network **A**. The Raznitsa coil in the dross of the galvanometer, where repeated closures did not surpass 0.5 divisions.

The closures for coil **A** were on the order of several tens of mA. For more of a quiescent operation the last detector has been glued with a thin layer of gum adhesive to a stone tile for support. Thus the mechanical period of the detector has been made very much greater in comparison with the period of parasitic mechanical fluctuations caused by the rotations of the rotor.

Table 1			
t x 10^{-4} sec		Exper	Theor
A	0,0	-	34,40
I	5,2	2,2	22,30
II	10,4	14,5	14,20
III	15,6	9,0	9,10
IV	20,8	6,4	5,80
V	26,0	4,0	3,73
VI	31,2	2,5	2,40
VII	36,4	1,5	1,54
VIII	41,6	-	0,90

The current induced in the disk is also induced in coils I through VIII, the magnitude of which was recorded by the detector (after rectification) by the maximum deflection of the ballistic galvanometer **Γ**. The individual galvanometer results were taken with the various coils, beginning with I and continuing on up through VII inclusive, and for each coil a maximum ballistic deflection recording was separately made. Hence measurements were completed by turns for each coil, beginning with I and on up through VII, via the detector on the galvanometer **Γ**, while a separate impulse in coil **A** moved. Having yielded some times for all the measurement series, we have received a maximum deflection value for them averaged together, as well as for each coil separately. The observed data is given in Table 1. In figure 7, the diagram of signal attenuation current in the disk over time is presented, and our experimentation shows that transit time from one point on one coil to another point on a separate coil 5.2 X 10^{-4} seconds. This resulting curve coincides with an exponential curve, and the relaxation time is equaled as t = 11.7 X 10^{-4} seconds. The value of an ordinate corresponding to a zero abscissa is calculated under the formula $I = I_0 e^{-jwt}$. Of the curve that is visible, the fulfillment by one turn of the disk, the current falls by 2.5% of its initial size. In the third tabulation column of Table 1, the calculated values of the function are given.

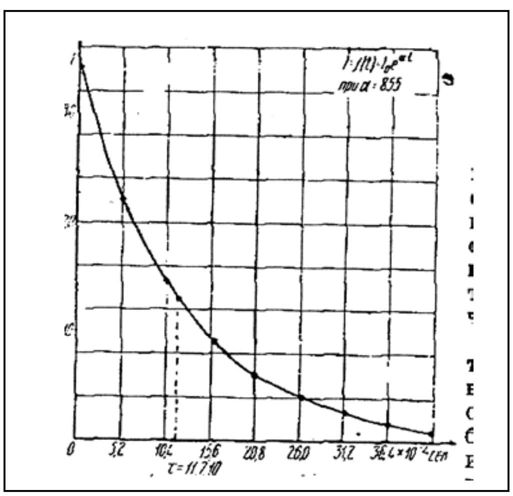

Figure 7: Curve of a waning magnetic field in a rotating disk

Before providing results of comparing Eq. (3) with experience, it is necessary to stop on one phenomenon that has influence on the depth of modulation and consequently a requirement for self-excitation of these Foucault bias currents in both moving as well as conducting mediums. The providing of a bias field for a rotating disk consists of the following: Let some variable field with a cross-sectional area **S** that induces an electric current in an immobile conductor. The topography of the induced current, especially in metal, that it will capture a magnetic field while "getting into" the metal. As a result, a primary field is erected between "the electrical coil," and the self-induction coil. Some mutual induction will define a response of a continuous conductor on a coil, i.e. on it self-inductance and ohmic resistance. The question is asked whether there will be a reaction of a continuous conductor to the coil will be the same if that conductor results in a forward rotary movement, i.e. whether the theory will underestimate the field generated by an electric coil in the same position concerning the coil, or if the driving movements of the conductor will drift. If drift exists, the common reactance should decrease, the self-inductance of the coil to increase, and ohmic resistance to decrease. For experimental checks of this phenomenon, we used the same adjustment for all of the received data described above. The machine's stators were joined to the Vitstona bridge. The current in the bridge when the disk was immobilized was compensated for, then the disk was activated. Already at low revolutions the neutralization was broken. As recorded in Table 2 the observed data is given.

Table 2

State of Disk	L Stators	ΔL	$\frac{\Delta L}{L}100$
Moving	0,0342	0,0008	3,2%
Unmoving	0,0250		
Moving	0,0242	0,0008	3,2%
Unmoving	0,0250		
Moving	0,02425	0,00075	3,1%
Unmoving	0,0250		
Moving	0,0253		
Spread of Uncertainty			3,1% +/-0,1%

From the table, we can see that there exists a bias current offset in the disk with relation to the coil, but the numbers are insignificant, not exceeding 3%, therefore with this data amendment we shall not consider matter further.

The Effects of Depths of Modulation on a Self-Inductance

For a picture of the pattern of system changes relating to self-induction that has dependence on disk response, with self-inducting stators positioned on the disk has been measured. This is how the curves in Figure 8 were gained. Here curve I gives the change in self-inductance of the stators at the angle of rotation of the disk. Curve II represents the course of change of the windage losses in the stators from the same argument. Minimum R corresponds to a tooth position between the coils (position A). Instead of one maximum which was natural to expect at a tooth position against the coil (position B). The curve has two maxima located on both legs from standing position (B). As has been shown by the special examinations yielded by N.F. Alekseev, the occurrences of these maxima are required by the origin of the original "edge effect." These regional effects consist of Foucault's currents that are induced continuously in the conductors, and then settle down in them in such a manner that they always cover the induced floor. Therefore, when the tooth on the disk has any edge that enters the magnetic field, the induced currents "are obliged" to settle down along this edge, irrespective of the quantity of teeth. And all the resistance to the induced current will be mainly concentrated at [along] this edge. In the process of removal of the tooth edge from the magnetic field, the resistance will decrease. As the field shape of the induced current will vary with the change in position of the particular standing tooth, it is obvious that the self-induction of the tooth will vary also. Thus we have a system consisting of primary and secondary windings, with a variable coupling coefficient and a self-induction, and the resistance of the secondary winding depending on the size of the coefficient. As for windage losses, we brought a tooth near a stator winding that was incremented with an increase in the ratio of the coupling coefficient and was defined in the given equivalent quantities by the formula:

$$R = \frac{M^2 \omega^2 R_1}{Z^2} \qquad (16)$$

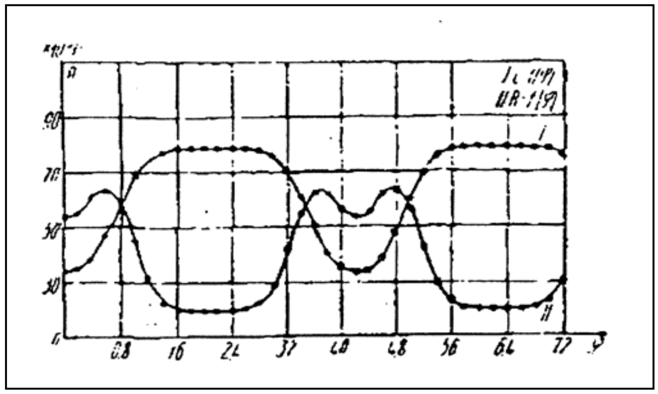

Figure 8: Ideal dependence on self-inductions (L) and resistances (R) of the system from a standing disk

Where **R**—brought losses, **M**—a common reactance “between” the coil and the tooth, **Z**—“impedance” of a tooth, and **R₁** wattage resistance of the tooth at the “occurrence” of a tooth in the floor as losses grow very promptly as together with the growth of **M**, we have major resistance $\mathbf{R_1}$. But even with the slightest chance of the further occurrence of the growth of **M** based on a tooth in the floor, coming nearer to the maximum, all will be slowed down, and resistance $\mathbf{R_1}$ will decrease. Therefore insertion losses will fall. At the output of a tooth, the field pattern will repeat. We shall not dally more on this or related matters very much as there are interesting phenomena occurring in continuous conductors (a special clause will be devoted to these questions), and we shall restrict ourselves only to those factors that occur without them, as fully describing the phenomenon of hetero-parametric excitation would be impossible. Below, we give the observed data (see Table 3). The depth of modulation T was defined, how

Table 3

	m	δ_{max}	$\frac{2}{\pi}\delta_{max}$
I	0,144	0,23	0,146
II	0,35	0,57	0,365
III	0,384	0,595	0,375

$$m = \frac{A_1}{\frac{1}{T}\int_0^T L(t)dt} \qquad (17)$$

Here A_1—amplitude of the harmonic corresponding frequency of modulation in a Fourier analysis of expression for self-induction $L(t) \rightarrow T$ period of modulation L(t)—a curve of self-induction, presented in Figure 8. Windage losses in the stators are stated as

$$R_0 = \frac{1}{T}\int_0^T R(t)dt \qquad (18)$$

Where R(t) – the dependence of the losses from a disk's angle of rotation, i. e., from time. Breaking down the L(t) gives the following expression:

$$L(t) = 0{,}0623 + 24\sin(\omega t - 86^\circ) - 7{,}16\sin(2\omega t - 67^\circ) + 1{,}17\sin(3\omega t + 67^\circ) + 1{,}17\cos 4\omega t + 0{,}5\cos 5\omega t.$$

In Table 3, values are given for $\delta_{\max}$ and $\frac{2}{\pi}\delta_{\max}$ for three cases.

From this table it is easily seen that the basic condition (3) for the excitation of hetero-parametric oscillations is always executed, but there is some discrepancy between corresponding values of the first and third columns, and we say that is not being considered by us, as it has already been noted elsewhere, due to the influence of the disk's driving motion.

Increasing Oscillations

Increasing oscillations at hetero-parametric excitation has a much more complex character, than in ordinary linear systems (with constant parameters). Using the approaches and methods specified by A. Mandelstam and N. D. Papaleski,[5]

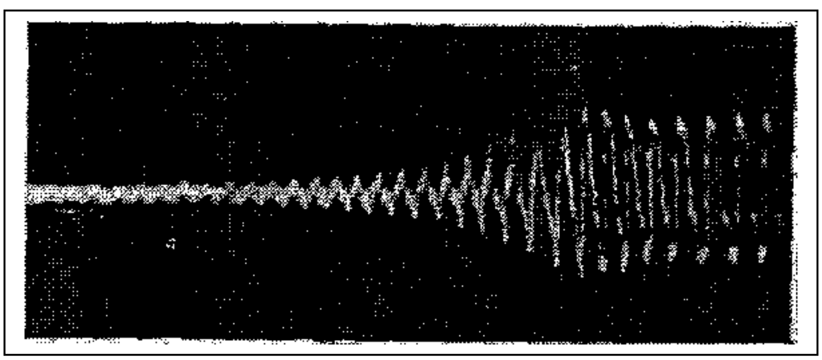

Figure 9

it is possible to reduce the problem to a system of two linear differential equations of the first order, which do not contain time explicitly. This circumstance presents greater advantages as it allows us to apply Poincare's methods of examination to the properties of integrated curves on the phase plane. Acting thus, V.P. Guliaev and V.V. Migulin[6] have analyzed qualitatively the character of increasing tightening at hetero-parametric excitation. We have assigned ourselves the task to experimentally approach this problem

[5]Loc. cit.

[6] V.P. Guliaev and V.V. Migulin, Loc. cit.

to clearing up the basic features of this process increase, together with the falling off of oscillations. With this purpose, we make a photographic entry for different decrements δ and m = constant. From the oscillograms in Figures 9 and 10 it is easy to see that the speed of increase incrementally increases with the [magnification?] of the difference.

$$\frac{\pi}{2}m - \delta = \xi \qquad (19)$$

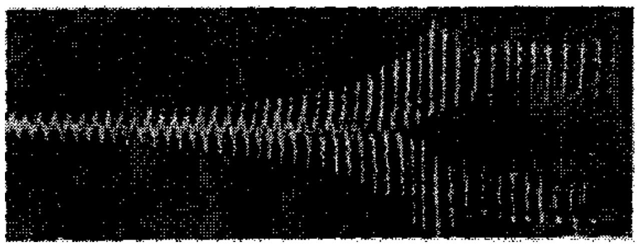

Figure 10: Oscillogram of the increasing oscillations (L) with hetero-parametric excitation. ξ = 0.415

Describing, as it is specified above Eq. (8) process of increasing at $\xi = 0$. Here ξ — is an incremental increase of the system. As the enveloping increase shows, that your contour already at insignificant current levels represents itself as a nonlinear system. The full time [word here?] of increase, displayed on the oscillogram in Figure 10, is almost two times the level as the highest level of time in oscillogram in Figure 9. An increment in the first case $\xi_1 = 0.11$; in the second case of a torus, $\xi_2 = 0.201$. Here certainly, it makes sense to speak about incrementation, although only during the moment of excitation as at further increasing oscillations it changes in size. To record the increase an oscillogram was taken by a loop on an immobile photographic plate. The fluctuations were developed. One mirror was strengthened on one axis to the direct current motor. The natural frequency was 12,000 revolutions, with a peak current of 100 mA. With the arc light source such an installation allows us to log data, without special work, for individual processes of a duration of 10^{-4} seconds, at a scanning rate of 150 m/second, i.e., by the process duration of 10^{-4} seconds that can be stretched on a time axis by 1.5 cm.

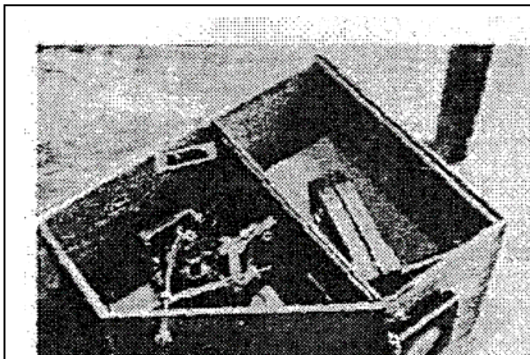

Figure 11

In Figure 11 the device interior where the photoregistry is can be seen.

For the photowriting processes of increase and decrease has been made near the switch, with the quickness with which it was possible to activate the system and that moment when the field entered the area of the sensitive stratum plate, with the simultaneous over-current operation of the mechanical-optical shutter. As the speed of the shutter stole up, its opening time was not less than the time of the opening of the shutter along the entire length of the plate, and no more than half the period of a revolution of the motor.

For a check of the correctness of the definition of the logarithmic decrement of the signal attenuation of the parameter measurements, the average values have been removed from the signal attenuations of the oscillograms. In Figures 12 and 13 the signal attenuation decrements of the oscillograms are shown in corresponding decrements of 0.414 and 0.555 respectively. From the start both oscillograms have expressed nonlinear character, but already after two periods of signal attenuation. A further falling off occurs under the exponential law. On a site where the curve is exponentially immediate, we can directly from the oscillogram define the logarithmic decrements. In Figure 12 for the decrement, we have found that the oscillogram's datum is δ_1= 0.400 rather than 0.414, and from Figure 13 we have found that δ_2= 0.600, rather than 0.555. With this result anyway it is

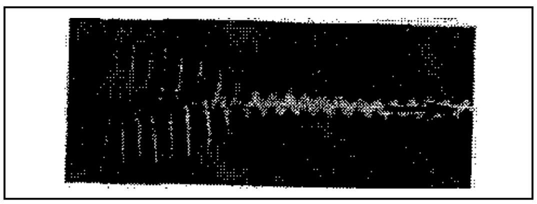

Figure 12: Logarithmic Decrement δ_1=0.414

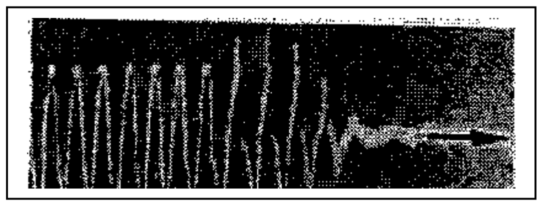

Figure 13: Logarithmic Decrement δ_2=0,555

necessary to consider it satisfactory. [For an effort at] greater concurrence to receive the data would be difficult, definition 3 is direct from the oscillogram, and without special tools does not provide for adequate accuracy. In summary we shall note that the excitation oscillations and their termination in the contour was yielded by a corresponding detuning, i.e., an adjustment of some cutout capacity. The oscillograms in Figures 12 and 13 have had such modes removed when the oscillations were broken by unplanned testing adventures at the corresponding capacity. On one of them (Figure 13) the unplanned tests revealed that the import capacity of the detuning the amplitude of the first 1.5 to two periods has to increase and only then signal attenuation begins. On the oscillogram in Figure 12 these phenomena are not observed. The explanation of this effect will be given in the following paragraph.

Establishment of a Stationary Mode

On a number of questions about the occurrence of oscillations via hetero-parametric excitations, the specific question about how to achieve a stationary (constant, not increasing or decreasing) amplitude is extremely important also. It has been specified in our introduction, that the necessary restriction on the increase of the amplitude of the oscillations is achieved by the presence of nonlinear parameters in the system. In the present research the "nonlinearity" is the nonlinear dependence in the self-induction that was caused by either the iron cores of the stator coils, or by the iron core of a special choke inserted into the system. As this choke had been formed by a transformer with two windings, in this last case it was the direct current through the other winding, and this will change the bias of the core to an index point on a magnetic hysteresis curve, and that it will at most change the character of the nonlinearity (coefficient α and γ in Eq. 9_1 and 10_1).

As Eq. 12 and 13 have shown, when we changed simultaneously μ and "detuning" ξ we can change the quantity of the stationary amplitude, so let's shift our attention to describing the operational materials. In dependence on whether the character of the nonlinearity of the system by iron stators was defined or by the special choke that we have essentially gained excellent results from each of the other two modes. One is the regime when the amplitude grows with the dimunition of the natural frequency of the contour, while the other—on the contrary, when the amplitude grows with an increase in the natural frequency. In Figures 14 and 15 both types are presented in comparison. On the abscissa axis the changes in capacity are postponed. On an axis ordinate—the effective amplitude values of the current go to ones dependent upon the voltages. Both cases prove to be true in reality, and also theoretically.

Table 4

No. exper.	m	δ	$2\xi_B$	$2\xi_M$
19r	0,35	0,414	26,3	34,0
19h	0,35	0,47	23,6	18,5
19	0,386	0,51	31,9	21,8

In Table 4, the comparative effects are presented via both experience and based on the calculated basis of the formula are presented for the peak detuning

$$2\xi = 2\sqrt{\frac{m^2}{4} - \frac{\delta^2}{\pi^2}}$$

gained from Table (4).
In Figure 15 it is immediately obvious from the curve data that I [via] detuning it has gained 2 ξ = 31.6%. However, obviously in the expressed tightening of a waning amplitude aside from the high frequencies here takes place. The corrected breadth strips = 26.3%. The satisfactory agreement with the original calculation is gained. Curve II immediately gives a detuning of 22.6%. Here with the divergence of nonlinearities the calculation is significant. But in this case the factor tightenings in this mode undoubtedly takes place, but we are not going to be considering it. At last on Figure 14 we already have significant divergence. Here we have two factors influencing a mistake: 1) under the influence of a load from a 3-phase motor, a current reduced the speed of rotation by 7.6%; 2) the greater detuning due to tightenings in this case was gained. The allowance of the motor was considered by us, as to the second allowance, without special complex measurements, we cannot define it.

Concerning all three cases it is possible to tell the following:

The detuning gained by practical consideration, will always be more than calculated by Eq. (7) does, because it does not consider absolute tightenings.

We have already previously discussed that we had gained two stationary modes, differing by that one nonlinearity is mostly caused by stators (a curve in Figure 14), while the other is gained by the means of a special choke that was consistently included in a spark circuit (the curve in Figure 15). In both cases it is possible to gain at a corresponding detuning very much smaller [instances] of stationary amplitude. The amplitude increases with a change of detuning in a determinate direction and almost springs [impinges] up to zero with further detuning. It is necessary to thus sweep aside that in the case of amplitude regulation by the means of a choke and biasing, the amplitude grows with

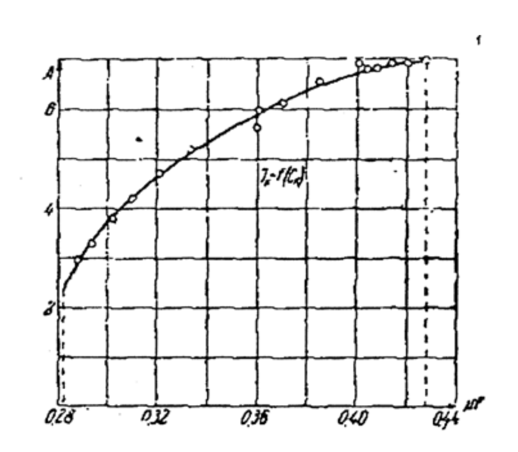

Figure 14: Dependence on stationary amplitude of a current (I) from a detuning (ξ)

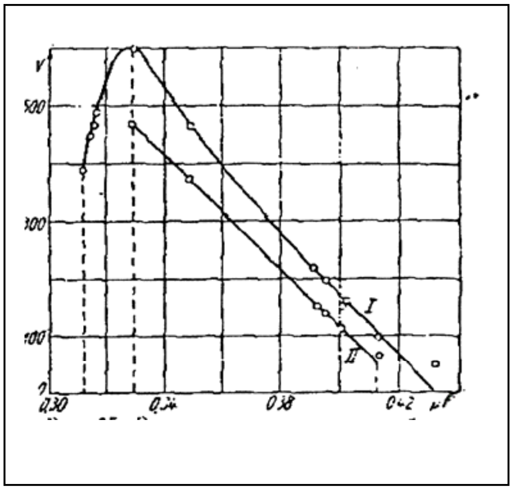

Figure 15: Dependence of stationary stress amplitudes (v) from a detuning contour (ξ)

$$
\begin{aligned}
&I \quad v_k = f(Ck) \qquad i_{pr} \approx 40mA \\
&\qquad \delta_I = 0{,}414 \\
&II \quad r_k = f(Ck) \\
&\qquad \delta_{II} = 0{,}47
\end{aligned}
$$

magnification of the natural frequency of the system. Setting the quantity of the initial biasing, it is possible to come up with any stationary amplitude. In the curves featured in Figures 16 and 17 the dependence of the current and of the voltage of the spark circuit depending on the biasing current is shown. The curve in Figure 16 is removed at a lower frequency than what is shown in Figure 17. At the same time, the electrical currents of the biasing curve of Figure 17 lie above as it corresponds to a greater natural frequency of the contour. It [also] speaks to the distinction [between] Figures 12 and 13 [where elements] correspond to the mode when a choke is included in the loop.

Let's note, as expressed in the theory given in Eq. (12), there are two similar modes which are defined by dependence on the magnetic flux of a current ($\gamma < 0$ or $\gamma > 0$).

In summary we shall note the following. As it has been previously specified, the essential role of the establishment of a stationary amplitude was carried out with the introduction into the system of a choke with biasing which worked upon the principal of the non-linear dependence of the magnetic flux in the choke's core due to a bias current in it. To make some representation of the quantitative leg of the phenomenon, we had measured dependences of a self-induction, as well as a choke and the windage losses in it from a biased current as given in Figure 18.

Table 5					
H_0	H	ΔR Ω	ΔL X 10^{-4} seconds	ΔR %	$\frac{\Delta L}{L_0}$ %
40n	5,6n	0,4	1,0	1,0	0,15
60n	18,2n	0,6	4,2	1,2	0,7
80n	37,8n	1,7	7,5	3,4	1,25

These curves allow us to define the quantity of change of the median value of self-induction (ΔL) and signal attenuations (ΔR) in a system at various amplitudes (H) of parametric oscillations and at various values

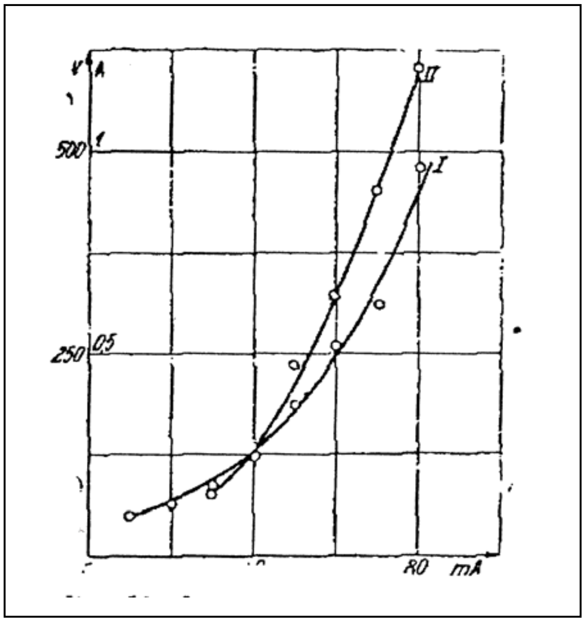

Figure 16: Dependence quantities [as in regards to] a stationary amplitude of a current (i) and voltages (v) from a biasing current of a choke (I)

$$\begin{aligned} &I \quad v = f_1(i), \\ &II \quad I = f_2(i), \\ &C_{const} = 0{,}375\,pF \end{aligned}$$

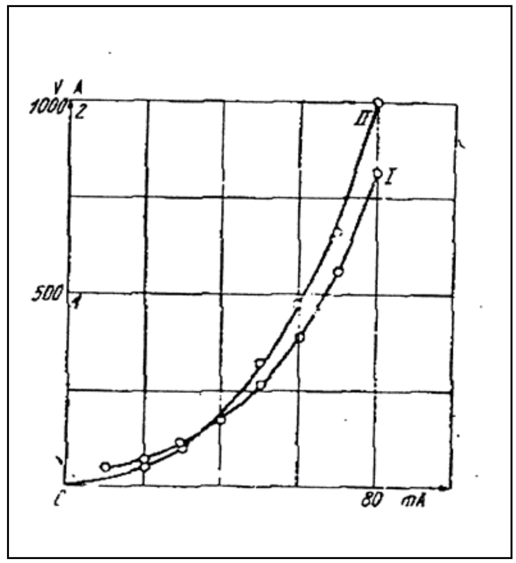

Figure 17: Dependence of a quantity of station amplitude (a current I and voltages v) from a biasing current (i)

$$\begin{aligned} &I \quad v = f_1(i), \\ &II \quad I = f_2(i), \\ &C_{cont} = 0{,}32\,pF \end{aligned}$$

of biasing current (H_0). As it appears, the data quantities from Table 5, practically $\frac{\Delta L}{L_0}$ *and* $\frac{\Delta R}{R_0}$, and it is on the order of less than 1-2%. To create such median values of a self-induction (and consequently, a median "natural" frequency) and the signal attenuations of the corresponding stationary amplitude oscillations, differ from the initial values of these quantities (at perpetually small values of amplitude) vary little. From here, inevitability streams, but it is impossible to explain the physical establishment process of a stationary amplitude with a gradual dimunition of an increment increase up to zero, owing to the change of frequency and the signal attenuation, caused by non-linear dependences in the system. As the theory proves, this process is related to system occurrences owing to the presence of diverging nonlinearities, the reactions amplifying the process of amplitude increase and aspiring to compensate for the action of the parameter modulation.

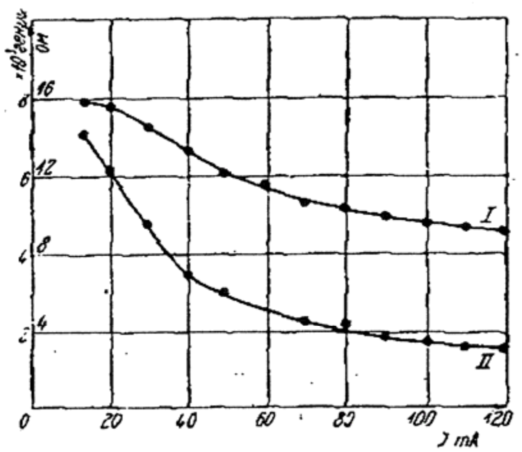

Figure 18

The Analysis of the Oscillations

Until now we viewed the vibrational process gained as a result of hetero-parametric excitation, as presented by one harmonic given by the □ zero□ solution of the differential equation to the problem. At small depths of modulation of the nonlinear parameter (10-15%) this solution as shown from practical experience, really transmits with sufficient approach the entire process. Except for basic frequency harmonics, half in comparison with the modulation frequencies, other harmonics are practically missed. At greater depths of modulation, however, in a basic series according to the theory should play a role and have higher harmonics. To realize the structure and intensity of these harmonics, and to also make for itself some representation about the influencing factors on the raising of the depth of modulation up to the point where the phenomena and other harmonics appear, a series of experiments were carried out and a number of oscillograms taken. The analysis of these oscillograms, two of which are presented as examples in Figures 19 and 20, show that at a depth of modulation of about 35%, already at rather small currents the contour curve of the current has a rather complex character, and with the oscillogram in Figure 19 when it was taken at a current in the bias loop of 4 amperes, the stationary amplitude was adjusted by changing the load in the form of a series of incandescent lamps.

I	$L_R = f_1(I)$
II	$R_R = f_2(I).$

Figure 19: Oscillogram where i=4 amperes

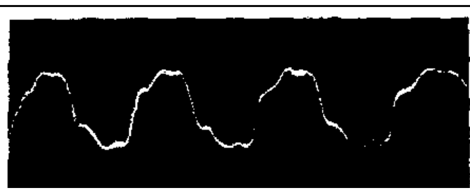

Figure 20: Oscillogram where i=8 amperes

Breaking down this curve into a Fourier series, we have discovered the following expression for i:

$$i = \sin \omega t + 0{,}387\sin\left(3\omega t + 9{,}5^{\circ}\right) + 0{,}209\sin\left(5\omega t + 142^{\circ}\right) + 0{,}00072\sin\left(7\omega t + 324{,}5^{\circ}\right) +$$
$$+ 0{,}016\sin\left(9\omega t + 146{,}4^{\circ}\right) + 0{,}00065\sin\left(11\omega t + 316{,}9^{\circ}\right).$$

Here it is interesting to note a lack of harmonics. The expansion of the oscillogram in Figure 20, which have had a contour taken in a loop at 8 amperes, gives for i:

$$i = \sin\left(\omega t + 81^{\circ}\right) + 0{,}121\sin\left(3\omega t - 17^{\circ}\right) + 0{,}165\sin\left(5\omega t - 13^{\circ}\right) + 0{,}041\sin\left(7\omega t + 38^{\circ}\right) +$$
$$+ 0{,}012\sin\left(9\omega t + 44^{\circ}\right) + 0{,}0053\sin\left(11\omega t - 63^{\circ}\right).$$

We see, that in this case we have varied the relative intensity of the odd numbered harmonics.

A more detailed inspection of this problem (both experimental and theoretical) will be given in another article.

This paper is related to a cycle of work that had a common origin in the research of Academician L.I. Mandelstam and Professor N. D. Papaleski, which has been made only in the last two years at the Leningrad Electro-Physical Institute, Laboratory of Nonlinear Systems. Additionally, I recognize Professor N. D. Papaleski, and I consider it an honored duty to express profound gratitude for his valuable suggestions on the present work.

Leningrad Electrophys. Inst., July, 1933 edited August 22, 1933

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"On the Excitation of Oscillations by Parameter Variation (Hetero-Parametric Excitation)"
By W.A. Lazerev

This work contains an experimental investigation of the conditions for the existence of oscillations in an electrical circuit (caused by mechanical vibrations periodically changing the self induction) as well as for the amplification of the same. In this case, an alternating current generator, constructed according to the principle given by L.I. Mendelstam and N.D. Papalexi, is described and investigated. The results of the experiments, both qualitative and quantitative, are consistent with the theory developed by the authors mentioned above.